ON THE COMPLEMENTED SUBSPACES PROBLEM

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ABSTRACT

A Banach space is isomorphic to a Hilbert space provided every closed subspace is complemented. A conditionally σ-complete Banach lattice is isomorphic to an L_p-space ($1 \leq p < \infty$) or to $c_0(\Gamma)$ if every closed sublattice is complemented.

The first theorem stated in the abstract has long been conjectured by various authors. The proof which we give here is surprisingly simple.

First, let us introduce a few notations. By l_2^n we denote the n-dimensional Hilbert space. The distance coefficient $d(X,Y)$ between two isomorphic Banach spaces X and Y will be defined as $\inf \| T \| \, \| T^{-1} \|$ where the infimum is taken over all possible isomorphisms T from X onto Y.

The proof of the theorem is based on the following three known results which we state here as propositions.

PROPOSITION 1. [2, p. 156]. *Let X be an infinite-dimensional Banach space. Then for every integer n and every $\varepsilon > 0$ there exists an n-dimensional subspace C of X such that $d(C, l_2^n) \leqq 1 + \varepsilon$.*

PROPOSITION 2. [1]. *Let X be a Banach space every subspace of which is complemented. Then there exists a $1 \leqq \lambda < \infty$ such that every finite-dimensional subspace B of X is the range of a projection of norm less than λ.*

PROPOSITION 3. [3], [6], [7]. *Let X be a Banach space for which there exists a constant $\beta < \infty$ such that for every finite-dimensional subspace B of X, $d(B, l_2^n) \leqq \beta$ (where $\dim B = n$). Then X is isomorphic to a Hilbert space.*

We shall now prove our main result.

THEOREM 1. *Assume that every closed subspace of a Banach space X is complemented. Then X is isomorphic to a Hilbert space.*

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PROOF. We will assume that X is an infinite-dimensional space, otherwise the assertion is trivial. Let λ be the constant whose existence is ensured by Proposition 2. Consider a finite-dimensional subspace B of X and let

$$\alpha = d(B, l_2^n) \tag{1}$$

where $\dim B = n$. Let Q be a projection from X onto B with $\| Q \| \leqq \lambda$. By Proposition 1 there is a subspace C of $(I - Q)X$ such that

$$d(C, l_2^n) \leqq 2. \tag{2}$$

It follows from (1) and (2) that there exists an operator T from B onto C such that

$$\| x \|/2\alpha \leqq \| Tx \| \leqq \| x \| ; \qquad (x \in B). \tag{3}$$

Let P be a projection of norm less than λ from X onto the subspace

$$D = \{x + \mu Tx \,|\, x \in B\}$$

where $\mu = 2^6\lambda^2$. If $V: B \to B$ is the operator defined by

$$PTx = Vx + \mu TVx; \qquad (x \in B) \tag{4}$$

then $Vx = QPTx$; $(x \in B)$ which implies

$$\| Vx \| \leqq \lambda^2 \| Tx \|. \tag{5}$$

Furthermore, notice that for each $x \in B$ we have

$$\begin{aligned} Px &= P(x + \mu Tx) - \mu PTx = x + \mu Tx - \mu(Vx + \mu TVx) \\ &= [x - \mu Vx] + \mu[Tx - \mu TVx]. \end{aligned}$$

Hence

$$(I - Q)Px = \mu[Tx - \mu TVx]. \tag{6}$$

Combining (3), (5) and (6) we obtain

$$\begin{aligned} \| (I - Q)Px \| &= \mu \| T(x - \mu Vx) \| \geqq \frac{\mu}{2\alpha} \| x - \mu Vx \| \\ &\geqq \frac{\mu}{2\alpha}(\| x \| - \mu \| Vx \|) \geqq \frac{\mu}{2\alpha}(\| x \| - \mu\lambda^2 \| Tx \|). \end{aligned} \tag{7}$$

Using (2) one can find an operator U from C onto l_2^n such that

$$\| y \|/2 \leqq \| Uy \| \leqq \| y \| ; \qquad (y \in C). \tag{8}$$

Let $\hat{T}$ be the operator from B into the $2n$-dimensional Hilbert space $l_2^n \oplus_2 l_2^n$ defined by

$$\hat{T}x = (UTx/2,\ U(I-Q)Px/4\lambda^2) \tag{9}$$

Then

$$\begin{aligned} \|\hat{T}x\| &\leqq \|U\|\,\|T\|\,\|x\|/2 + \|U\|\,\|I-Q\|\,\|P\|\,\|x\|/4\lambda^2 \\ &\leqq \|x\|/2 + \lambda(\lambda+1)\|x\|/4\lambda^2 \leqq \|x\|. \end{aligned} \tag{10}$$

On the other hand by (7), (8) and (9) we have

$$\begin{aligned} \|\hat{T}x\| &\geqq \max(\|UTx\|/2,\ \|U(I-Q)Px\|/4\lambda^2) \\ &\geqq \max(\|Tx\|/4,\ \|(I-Q)Px\|/8\lambda^2) \\ &\geqq \max\Big(\|Tx\|/4, \frac{\mu}{16\alpha\lambda^2}(\|x\| - \mu\lambda^2\|Tx\|)\Big). \end{aligned} \tag{11}$$

We shall distinguish between two cases. If $\lambda^4 2^7 \|Tx\| \geqq \|x\|$ then

$$\|\hat{T}x\| \geqq \|Tx\|/4 \geqq \|x\|/\lambda^4 2^9. \tag{12}$$

Otherwise, i.e. if $\lambda^4 2^7 \|Tx\| < \|x\|$, then since $\mu = 2^6\lambda^2$ we have

$$\|\hat{T}x\| \geqq \frac{\mu}{16\alpha\lambda^2}(\|x\| - \lambda^4 2^6 \|Tx\|) \geqq \frac{\mu}{2^5\alpha\lambda^2}\|x\| = \frac{2}{\alpha}\|x\|. \tag{13}$$

Thus in either case

$$\|\hat{T}x\| \geqq \|x\| \min\left(\frac{2}{\alpha}, \frac{1}{\lambda^4 2^9}\right); \qquad (x \in B) \tag{14}$$

i.e. $\hat{T}$ is an isomorphism from B onto l_2^n. Since α is minimal it follows immediately from (1), (10) and (14) that $\alpha \leqq \lambda^4 2^9$ and the proof can be now completed by using Proposition 3.*

Additional results. A. Grothendieck [4, Chap. II, p. 73] who, among others, conjectured the validity of Theorem 1, has shown that it implies the following result.

THEOREM 2. *The only separable infinite-dimensional Frechet spaces in which every closed subspace is complemented are: s, $s \oplus l_2$, and l_2, where s is the space of all sequences of reals (i.e. the product of a sequence of lines).*

The proof of Theorem 1 can also be used to show that the following holds.

THEOREM 3. *Let X be a Banach space such that for every closed subspace Y of X and every compact operator $U : Y \to Y$ there exists a bounded linear extension $\hat{U} : X \to Y$ of U. Then X is isomorphic to a Hilbert space.*

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* A little more careful calculation shows even that $\alpha \leq 16\,\lambda^2$.

Indeed, one can easily see that a slight change of the proof of Proposition 2 in [1] under the assumptions of Theorem 3 will imply the existence of a uniform bound λ for the projections on finite-dimensional subspaces as in Proposition 2.

Our next two theorems show that in a space with an unconditional basis or more generally in a conditionally σ-complete Banach lattice, the existence of projections on every closed sublattice or on other selected classes of closed subspaces already implies that the space is of L_p $(1 \leq p < \infty)$ or c_0 type. These results are sharper forms of theorems proved in [10] and their proofs require only slight changes in the arguments used in [10].

Let $\{x_n\}_{n=1}^{\infty}$ be a basis of a Banach space X and let $p_1 < p_2 < \cdots$ be an increasing sequence of integers. Any basis having the form

$$y_k = \sum_{n=p_k+1}^{p_{k+1}} \lambda_n x_n\,; \qquad (k = 1,2,\cdots)$$

with λ_n scalars is called a block basis of $\{x_n\}_{n=1}^{\infty}$.

THEOREM 4. *A Banach space X with an unconditional basis $\{x_n\}_{n=1}^{\infty}$ is isomorphic to either c_0 or to l_p $(1 \leq p < \infty)$ if (and only if) for every permutation π of the integers and every block basis $\{y_k\}_{k=1}^{\infty}$ of $\{x_{\pi(n)}\}_{n=1}^{\infty}$ there exists a projection in X whose range is the subspace generated by $\{y_k\}_{k=1}^{\infty}$.*

PROOF. First, we may assume with no loss of generality that the unconditional constant of $\{x_n\}_{n=1}^{\infty}$ is 1, i.e. $\left\| \sum_{i\in\sigma} \alpha_i x_i \right\| \leq \left\| \sum_{i=1}^{\infty} \alpha_i x_i \right\|$ for each set σ of integers and scalars α_i for which $\sum_{i=1}^{\infty} \alpha_i x_i$ is convergent. Let $\{u_k\}_{k=1}^{\infty}$ and $\{v_k\}_{k=1}^{\infty}$ be two normalized block bases of $\{x_{2n}\}_{n=1}^{\infty}$, respectively $\{x_{2n-1}\}_{n=1}^{\infty}$. We are going to show that $\{u_k\}_{k=1}^{\infty}$ and $\{v_k\}_{k=1}^{\infty}$ are equivalent unconditional bases. Indeed, suppose a series $\sum_{k=1}^{\infty} \alpha_k v_k$ converges while $\sum_{k=1}^{\infty} \alpha_k u_k$ is divergent. As in [10] we will consider two different cases.

Case I. The subspace $\mathrm{clm}\{u_k\}_{k=1}^{\infty}$ is weakly sequentially complete. If the series $\sum_{k=1}^{\infty} \alpha_k \eta_k u_k$ converges for each sequence $\{\eta_k\}_{k=1}^{\infty} \in c_0$ then $\sum_{k=1}^{\infty} \alpha_k u_k$ is weakly unconditionally convergent and thus $\mathrm{clm}\{u_k\}_{k=1}^{\infty}$ contains a subspace isomorphic to c_0 which contradicts our assumptions. Consequently, we can assume the existence of a sequence $\{\eta_k\}_{k=1}^{\infty} \in c_0$ with $\eta_k \geq 0$ $(k = 1,2,\cdots)$ such that $\sum_{k=1}^{\infty} \alpha_k \eta_k u_k$ is still divergent. Consider now the projection P whose range is the subspace generated by the block basis $\{\eta_k u_k + v_k\}_{k=1}^{\infty}$ (of a certain permutation of $\{x_n\}_{n=1}^{\infty}$). Then

$$\left.\begin{aligned} Pu_k &= \sum_{j=1}^{\infty} c_j^{(k)}(\eta_j u_j + v_j) \\ Pv_k &= \sum_{j=1}^{\infty} d_j^{(k)}(\eta_j u_j + v_j) \end{aligned}\right\} ; \qquad (k = 1,2,\cdots).$$

Since $\{x_n\}_{n=1}^{\infty}$ is an unconditional basis there exists a projection E of norm 1 such that $Ex_{2n} = x_{2n}$, $Ex_{2n-1} = 0$ $(n = 1,2,\cdots)$. Thus

$$EPv_k = \sum_{j=1}^{\infty} d_j^{(k)} \eta_j u_j ; \qquad (k = 1,2,\cdots).$$

Obviously EP can be considered as an operator from $\operatorname{clm}\{v_k\}_{k=1}^{\infty}$ into $\operatorname{clm}\{u_k\}_{k=1}^{\infty}$ which is defined by the infinite matrix $(d_j^{(k)} \eta_j)$; $(k,j = 1,2,\cdots)$. Since both $\{u_k\}_{k=1}^{\infty}$ and $\{v_k\}_{k=1}^{\infty}$ are unconditional bases of constant 1, it follows (cf. [9]) that the diagonal matrix defines an operator D: $\operatorname{clm}\{u_k\}_{k=1}^{\infty} \to \operatorname{clm}\{v_k\}_{k=1}^{\infty}$ such that $\| D \| \leqq \| EP \| \leqq \| P \|$. For this operator we have

$$Dv_k = d_k^{(k)} \eta_k u_k ; \qquad (k = 1,2,\cdots).$$

Thus the convergence of the series $\sum_{k=1}^{\infty} \alpha_k v_k$ implies that of the series

$$\sum_{k=1}^{\infty} \alpha_k d_k^{(k)} \eta_k u_k .$$

Let us also observe that

$$\left.\begin{aligned} \eta_k c_k^{(k)} + d_k^{(k)} &= 1 \\ |c_k^{(k)}| &\leqq \| P \| \end{aligned}\right\} ; \qquad (k = 1,2,\cdots)$$

and therefore $\lim_{k\to\infty} d_k^{(k)} = 1$. Hence $\sum_{k=1}^{\infty} \alpha_k \eta_k u_k$ converges, contrary to our hypothesis.

Case II. The subspace $\operatorname{clm}\{u_k\}_{k=1}^{\infty}$ is not weakly sequentially complete. Since $\{u_k\}_{k=1}^{\infty}$ is an unconditional basic sequence, there exists (cf. [5]) a block basis $\{w_j\}_{j=1}^{\infty}$; $\| w_j \| = 1$; $(j = 1,2,\cdots)$ of $\{u_k\}_{k=1}^{\infty}$ which is equivalent to the natural basis of c_0.

We shall show now that $\{v_k\}_{k=1}^{\infty}$ is equivalent to the usual basis of c_0. Indeed, if $\sum_{k=1}^{\infty} \beta_k v_k$ is a divergent series with $\lim_{k\to\infty} \beta_k = 0$ and Q is the projection whose range is the closed subspace generated by the block basis $\{|\beta_k|^{1/2} v_k + w_k\}_{k=1}^{\infty}$ then, by repeating the arguments already used in the previous case, we can show that the convergence of the series $\sum_{k=1}^{\infty} |\beta_k|^{1/2} w_k$ implies that of $\sum_{k=1}^{\infty} |\beta_k| v_k$, which is contradictory. Now, replacing $\{v_k\}_{k=1}^{\infty}$ by $\{u_k\}_{k=1}^{\infty}$, we can prove that

$\{u_k\}_{k=1}^{\infty}$ is also equivalent to the usual basis of c_0. That completes the proof of Case II.

In order to finish the proof of the theorem notice that we have just shown that both $\{x_{2n}\}_{n=1}^{\infty}$ and $\{x_{2n-1}\}_{n=1}^{\infty}$ are equivalent to each of their possible normalized block bases. Thus, by a result of M. Zippin [11], $\{x_{2n}\}_{n=1}^{\infty}$ as well as $\{x_{2n-1}\}_{n=1}^{\infty}$ are equivalent to the usual basis of c_0 or l_p for some p $(1 \leqq p < \infty)$ and so is $\{x_n\}_{n=1}^{\infty}$.

Zippin's result from [11] clearly remains valid even if one assumes a basis is equivalent to each normalized block basis with *non-negative* coefficients. Therefore, in the statement of Theorem 4 we can merely require the existence of projections on block bases with non-negative coefficients. The next isomorphic characterization of c_0 and L_p-spaces $(1 \leqq p < \infty)$ follows from this fact and [10, Propositions 6 and 7]. Let us just recall that a conditionally σ-complete Banach lattice is a vector lattice endowed with a norm ρ satisfying $\rho(u) \leqq \rho(v)$ if $|u| \leqq |v|$ and such that every order-bounded sequence has a least upper bound.

THEOREM 5. *A conditionally σ-complete Banach lattice L_ρ is isomorphic to either $c_0(\Gamma)$ for some abstract set Γ, or to L_p $(1 \leqq p < \infty)$ on some measure space, provided every closed sublattice is complemented.*

REMARKS. 1) This theorem has been proved in [10] under the additional assumption that the projections are positive.

2) Since L_p-spaces $(1 \leqq p \neq 2 < \infty)$ as well as $c_0(\Gamma)$ are known to have uncomplemented subspaces, Theorems 4 and 5 give an alternative proof of Theorem 1 in the case when X has an unconditional basis or is a conditionally σ-complete Banach lattice.

3) From Theorem 4 it is easy to deduce an alternative proof of the main result of Pelczynski and Singer [8]. Indeed, if X has an unconditional basis $\{x_n\}_{n=1}^{\infty}$ and if X is not l_p $(1 \leqq p < \infty)$ or c_0, then there is a normalized block basis of $\{x_{\pi(n)}\}_{n=1}^{\infty}$, for some permutation π, which spans an uncomplemented subspace. By a result of [12] this block basis can be extended to a basis of X which must be a conditional basis. If $X = l_p$ for some p or c_0, a conditional basis in X can be exhibited explicitly.

REFERENCES

1. W. J. Davis, D. W. Dean and I. Singer, *Complemented subspaces and Λ systems in Banach spaces*, Israel J. Math. 6 (1968), 303–309.

2. A. Dvoretzky, *Some results on convex bodies and Banach spaces*, Proc. Int. Symp. on Linear Spaces, Jerusalem, 1961, pp. 123–160.

3. A. Dvoretzky, *A characterization of Banach spaces isomorphic to inner product spaces,* Proc. Coll. Convexity, Univ. of Copenhagen, 1966, pp. 61–66.

4. A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires,* Mem. Amer. Math. Soc. **16** (1955).

5. R. C. James, *Bases and reflexivity of Banach spaces,* Ann. of Math. (2) **52** (1950), 518–527.

6. J.T. Joichi, *Normed linear spaces equivalent to inner product spaces,* Proc. Amer. Math. Soc. **17** (1966), 423–426.

7. J. Lindenstrauss, *On the modulus of smoothness and divergent series in Banach spaces,* Michigan Math. J. **10** (1963), 241–252.

8. A. Pelczynski and I. Singer, *On non-equivalent bases and conditional bases in Banach spaces,* Studia Math. **25** (1964), 5–25.

9. A. E. Tong, *Diagonal submatrices of matrix maps,* Pacific J. Math. **32** (1970), 551–559.

10. L. Tzafriri, *An isomorphic characterization of L_p and c_0–spaces.* II, to appear in Michigan Math J. **17** (1970).

11. M. Zippin, *On perfectly homogeneous bases in Banach spaces,* Israel J. Math. **4** (1966), 265–272.

12. M. Zippin, *A remark on bases and reflexivity in Banach spaces,* Israel J. Math. **6** (1968), 74–79.

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